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Fractal geometry: what is it, and what does it do?

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[Plate 1]

Fractal geometry is a workable geometric middle ground between the excessive geometric order of Euclid and the geometric chaos of general mathematics. It is based on a form of symmetry that had previously been underused, namely invariance under contraction or dilation. Fractal geometry is conveniently viewed as a language that has proven its value by its uses. Its uses in art and pure mathematics, being without 'practical' application, can be said to be poetic. Its uses in various areas of the study of materials and of other areas of engineering are examples of practical prose. Its uses in physical theory, especially in conjunction with the basic equations of mathematical physics, combine poetry and high prose. Several of the problems that fractal geometry tackles involve old mysteries, some of them already known to primitive man, others mentioned in the Bible, and others familiar to every landscape artist.

Fractals provide a workable new middle ground between the excessive geometric order of Euclid and the geometric chaos of roughness and fragmentation

Instead of attempting to introduce and link together the papers that follow in this Discussion Meeting, we prefer to ponder the question, 'What is fractal geometry?' We write primarily for the comparative novice, but have tried to include tidbits for the already informed reader.

Before we tackle what a *fractal* is, let us ponder what a fractal *is not*. Take a geometric shape and examine it in increasing detail. That is, take smaller and smaller portions near a point P, and allow every one to be dilated, that is, enlarged to some prescribed overall size. If our shape belongs to standard geometry, it is well known that the enlargements become increasingly smooth. Ultimately, nearly every connected shape is locally linear. One can say, for example, that 'a generic curve is attracted under dilations' to a straight line (thus defining the tangent at the point P). And 'a generic surface is attracted by dilation' to a plane (thus defining the tangent plane at the point P). More generally, one can say that nearly every standard shape's local structure converges under dilation to one of the small number of 'universal attractors'. The term 'attractor' is borrowed from dynamics and probability theory, and the even more grandiose term, 'universal', is borrowed from recent physics. An example of exception to this rule is when P is a double point of a curve; the curve near P is then attracted to two intersecting lines and has two tangents; but double points are few and far between in standard curves.

Standard geometry and calculus (which is intimately related to it) have long proven to be extraordinarily effective in the sciences. Yet there is no question that Nature *fails* to be locally linear. Indeed, the shapes of Nature are so varied as to deserve being called 'geometrically chaotic', unless proven otherwise. Unfortunately, 'complete' chaos could not conceivably lead to a science. This is perhaps why many of the oldest concerns of Man, such as concerns with the shapes of mountains, clouds and trees or, with the floods of the Nile, had not led to sciences comparable in effectiveness to the physics of smooth phenomena.

Though the term 'chaos' was not used, one can say that a second kind of chaos became known during the half century, 1875–1925, when mathematicians who were fleeing from concerns with Nature took cognizance of the fact that a geometry shape's roughness may conceivably *fail* to vanish as the examination becomes more searching. It may conceivably vary endlessly, up and down. The hold of standard geometry was so powerful, however, that the shapes constructed so that they do not reduce locally to straight lines were labelled 'monsters' and 'pathological'.

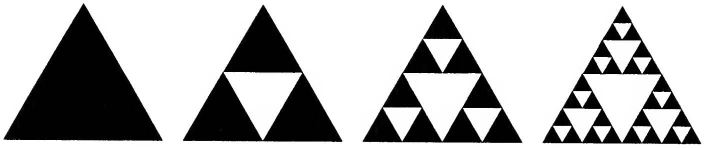
Between the extremes of linear geometric order and of geometric chaos ruled by 'pathologies', can there be a middle ground of 'organized' or 'orderly' geometric chaos? The author has conceived and outlined such a ground, and gave its study the name of 'fractal geometry', the fuller name being 'fractal geometry of nature and chaos'. It will be argued momentarily that fractal geometry is best viewed as a geometric language, new as of 1975, which incorporates as 'characters' several of the mathematical monsters of 1875–1925, and whose uses have now become so diverse, that it is possible to sort them out as poetry, strictly utilitarian prose and high prose.

Fractals are characterized by so-called 'symmetries', which are invariances under dilations and/or contractions

Broadly speaking, mathematical and natural fractals are shapes whose roughness and fragmentation *neither* tend to vanish, *nor* fluctuate up and down, but remain *essentially unchanged* as one zooms in continually and examination is refined. Hence, the structure of every piece holds the key to the whole structure.

An alternative term is 'self-similar', which has two meanings. One can understand 'similar' as a loose everyday synonym of 'analogous'. But there is also the strict textbook sense of 'contracting similarity'. It expresses that each part is a linear geometric reduction of the whole, with the same reduction ratios in all directions. Figure 1 illustrates a standard strictly self-similar fractal, called Sierpiński gasket. In the early days, the resulting terminological ambiguity was acceptable to physicists, because early detailed studies did indeed concentrate on strictly self-similar shapes. However, more recent developments have extended, in particular, to include self-affine shapes, in which the reductions are still linear but the reduction ratios in different directions are different. For example, a relief is nearly self-affine, in the sense that to go from a large piece to a small piece one must contract the horizontal and vertical coordinates in different ratios.

Can the process of taking parts be inverted, by replacing zooming in by zooming



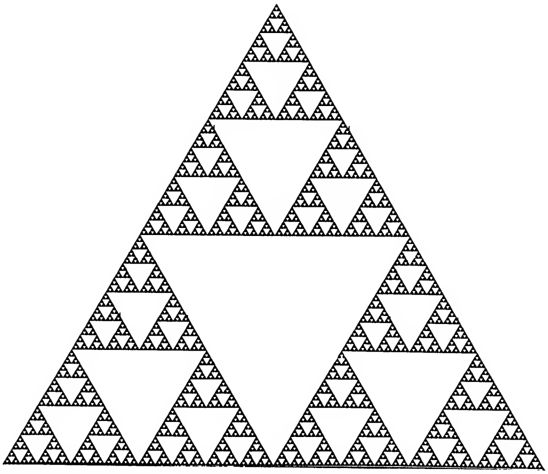


FIGURE 1. The Sierpiński gasket: construction and self-similarity properties, static and dynamic. The four small diagrams show the point of departure of the construction, then its first three stages, while the large diagram shows an advanced stage. The basic step of the construction is to divide a given (black) triangle into four sub triangles, and then erase (whiten) the middle fourth. This step is first performed with a wholly black filled-in triangle of side 1, then with three remaining triangles of side $\frac{1}{2}$, etc. Perform a similarity (or more precisely a homothety), that is, an isotropic linear reduction whose ratio is $\frac{1}{2}$, and whose fixed point is either of the three apexes of the triangle that circumscribes the gasket. It is obvious by examining the large advanced stage picture, that each of the three reduced gaskets is simply one third of the overall shape. For this reason, the fractal gasket is said to have three 'self-similarity' properties. As defined, these self-similarity properties seem 'static' and 'after-the-fact.' However, it is also possible to reinterpret them as forming together a generalized dynamical system. In that case, the gasket becomes redefined as a dynamic 'attractor'.

away from the object of interest? Indeed, many fractals, including the strictly self-similar ones, can be extrapolated to become unbounded, and they can remain unchanged when the above process is reversed and the shaped is examined at increasingly rough scales. To gain an idea of the appearance of an extrapolated Sierpiński gasket, it suffices to focus on a very small piece of figure 1, and then to imagine that this piece has been blown up to letter size, and therefore the whole gasket has become so large that its edges are beyond visibility. This illustration shows that each choice of a small piece to focus upon yields a different extrapolate.

When the Sierpiński gasket is constructed as in figure 1, that is, by deleting middle triangles, one sees it has three properties of contracting self-similarity, as pointed out in the caption of figure 1. These properties appear, so to speak, as 'static' and 'after-the-fact', but this is a completely misleading impression. Its

prevalence and its being viewed as a flaw came to us as a surprise. Therefore, it is good to stop and show how, knowing the same symmetries, it is easy to reconstruct the gasket 'dynamically' by a stochastic interpretation of a scheme due to J. Hutchinson. The basic principle of this scheme first arose long ago, in the work of Poincaré and Klein, and corresponding illustrations using randomization are found in our book (Mandelbrot 1982). Start with an 'initiator' that is an arbitrary bounded set, for example is a point P_0. Denote the three similarities of the gasket by S_0, S_1 and S_2, and denote by $k(m)$ a random sequence of the digits 0, 1 and 2. Then define an 'orbit' as made of the points $P_1 = S_{k(1)}(P_0)$, $P_2 = S_{k(2)}(P_1)$ and more generally $P_j = S_{k(j)}(P_{j-1})$. One finds that this orbit is 'attracted' to the gasket, and that after a few stages it describes its shape very well.

Surprise: simple rules can generate rich structure

How did fractals come to play their role of 'extracting order out of chaos?' To understand, one must go beyond simple shapes like the gasket or like the other fractals-to-be that mathematicians have first introduced as counter-examples. In these 'old' shapes, indeed, what one gets out follows easily from what has been knowingly put in. The key to fractal geometry's effectiveness resides in a very surprising discovery the author has made, largely thanks to computer graphics.

The algorithms that generate the other fractals are typically so extraordinarily short, as to look positively dumb. This means they must be called 'simple'. Their fractal outputs, to the contrary, often appear to involve structures of great richness.

A priori, one would have expected that the construction of complex shapes would necessitate complex rules. Thus, fractal geometry can be the study of geometric shapes that may seem chaotic, but are in fact perfectly orderly.

Let us, for the sake of contrast, comment on the examples of a related match between mathematics and the computer that arise in areas such as the study of water eddies and wakes. In these examples, the input in terms of reasoning or of number of lines of program is extremely complicated, perhaps more complicated even than the output. Therefore, one may argue that, overall, total complication does not increase in those examples, merely changes over from being purely conceptual to being partly visual. This change-over is very important and very interesting, but fractal geometry gives us something very different.

What is the special feature that makes fractal geometry perform in such unusual manner? The answer is very simple; the algorithm involves 'loops'. That is, the basic instructions are simple, and their effects can be followed easily. But let these simple instructions be followed repeatedly. Unless one deals with the simplest old fractals (Cantor set or Sierpiński gasket), the process of iteration effectively builds up an increasingly complicated transform, whose effects the mind can follow less and less easily. Eventually, one reaches something that is 'qualitatively' different from the original building block. One can say that the situation is a fulfilment of what is general is nothing but a dream: the hope of describing and explaining 'chaotic' nature as the cumulation of many simple steps.

Fractal geometry viewed as a language

'Mathematics *is* a language.' (Josiah Willard Gibbs (speaking at a Yale faculty meeting ... on elective course requirements).)

'Philosophy is written in this vast book – I mean the Universe – which stands forever open to our gaze, but cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geometrical figures, without which it is humanly impossible to understand a single word of it; without which one wanders in vain through a dark labyrinth.' (Galileo Galilei: *Il Saggiatore* (*The Assayer*) 1623).

'The language of mathematics reveals itself unreasonably effective in the natural sciences..., a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure even though perhaps also to our bafflement, to wide branches of learning.' (Eugene Wigner 1960).

Inspired by the above quotes, the best is to call fractal geometry a new *geometric language*, which is geared towards the study of diverse aspects of diverse objects, either mathematical or natural, that are not smooth, but rough and fragmented to the same degree at all scales. Its history is interesting and curious. As is clearly indicated by such terms as 'Cantor set', 'Peano curve', 'Sierpiński gasket', etc. ..., several fractal 'characters' date to 1875–1925. They count among the least complex (hence least beautiful), and they have seen previous uses in other languages that have nothing to do with fractal geometry.

However, as a language addressed to its new goals, fractal geometry was born with Mandelbrot (1975), the first edition of our book *Les objets fractals*.

There is a profound historical irony in the fact that these old 'characters' of the new geometry had been among the 'monsters' to which we have referred earlier. The general monster is *not* scaling, and the fact that some monsters are scaling had not been singled out, because it was viewed as a 'special' property, therefore one that is not very 'interesting'. In the mathematical culture of the century that ran from 1875 to 1975, special properties did not warrant investigation.

Clearly, the study of order in geometric chaos could not arise as a specialized object of study, and a term to denote it did not become indispensable, until we had performed two tasks. (i) We saw that diverse rough patterns of nature – noise and turbulence and diverse geographical features – are geometrically scaling. Eventually we, and now many others, have identified geometric scaling in many other areas of nature, and then explored its consequences. (ii) We saw that the proper tool to tackle scaling in nature is suggested by some of the old 'monsters'. The old monsters themselves are not realistic models, but the construction of new fractals was immediately spurred by the new need, and more accurate models soon became available.

A language can be appreciated in diverse ways. For fractal geometry, the reasons can be sorted into five categories: artistic, mathematical, 'historical', practical and scientific. Let us discuss them in turn.

The language of fractal geometry often creates plastic beauty

We use the term 'geometry' in a very archaic sense, as involving concrete actual images. Lagrange and Laplace once boasted of the absence of any pictures in their works, and their lead was eventually followed almost universally. Fractal geometry is a reaction against the tide, and a first reason to appreciate fractal geometry, because the 'characters' it adds to the 'alphabet' Galileo had inherited from Euclid, often happen to be intrinsically attractive. Many have promptly been accepted as works of a new form of art. Some are 'representational', in fact are surprisingly realistic 'forgeries' of mountains, clouds or trees, while others are totally unreal and abstract. Yet all strike almost everyone in forceful, almost sensual, fashion. The artist, the child and the 'man in the street' never seem to have seen enough, and they had never expected to receive anything of this sort from mathematics. Neither had the mathematician expected his field to interact with art in this way.

In any event, fractal geometry shows that there is an unexpected parallel to the above classic quote from Wigner. We have been fortunate to witness the revelation of the "*'unreasonable' and 'undeserved' effectiveness of mathematics as a source of enjoyable form.*"

Fractal geometry and mathematics

One can also appreciate the language of fractals because it happens to have led to new mathematics that several groups of specialists find attractive. To the layman, fractal art tends to seem simply magical, but no mathematician can fail to try and understand its structure and its meaning. The remarkable aspects of recent events is that much of this mathematics, had its origin been hidden, could have passed as 'pure', in other words, as absolutely self-referential.

Many of the early pictures of fractals have mostly served to 'visualize' facts that had been obtained previously by abstract 'pure mathematical' thought. For example, one who examines the computer-generated pictures of Julia sets and Fatou domains, which are mathematical objects that concern the dynamics of 'rational iteration', cannot help being filled with deep humility and boundless admiration for the creative powers Pierre Fatou and Gaston Julia had exhibited in 1917–19. He may also be tempted to echo Georg Cantor's assertion 'that the essence of mathematics resides in its freedom'. Such arrogance has indeed characterized much of mathematics during the period 1925–75, but an examination of further examples of fractal art necessarily brings it to an end. To many mathematicians, the newly opened possibility of playing with pictures interactively, has turned out to reveal a new mine of purely mathematical questions and of conjectures, of isolated problems and of whole theories. To take an example, examination of the Mandelbrot set leads to many conjectures that were simple to state, but then proved very hard to crack. To mathematicians, their being difficult and slow to develop does not make them any less fascinating, because a host of intrinsically interesting 'side-results' have been obtained in their study.

Herein hangs a tale. Pure mathematics does exist as one of the remarkable activities of Man, it certainly is different in spirit from the art of creating pictures by numerical manipulation, and it has indeed proven that it can thrive in splendid isolation, at least over some brief periods. Nevertheless, the interaction between art, mathematics and fractals confirms what is suggested by almost all earlier experiences. Over the long haul mathematics gains by not attempting to destroy the 'organic' unity that appears to exist between seemingly disparate but equally worthy activities of Man, the abstract and the intuitive.

Nevertheless, fractal geometry is not (at least as of today) a branch of mathematics like, for example, the theory of measure and integration. It fails to have a clean definition and unified tools, and it fails to be more or less self contained. Calling it 'fractal geometry of nature and of chaos', immediately explains why it is more like the theory of probability. Both thrive only when they are not over-defined, when they do not mind using mismatched tools, and when they overlap heavily with many neighbouring endeavours. Moreover, the comparison with probability is not meant to imply a comparable level of development. While the fractalists have some reason for modest pride, their mathematical language keeps continuing to evolve and to expand with each new use. To find that a new use requires the improvement of some basic point is not an exception, rather an everyday experience.

FRACTALS AND SOME GREAT OLD 'PROBLEMS THAT POSE THEMSELVES'

There is a third reason to appreciate fractal geometry, whose validity is totally independent of the earlier two. This new reason involves the antiquity and the prevalence of the natural phenomena on which it has made a dent, either by descriptions that suffice to the engineer or by theories that satisfy the sophisticate.

The great Henri Poincaré drew, long ago, a distinction between those problems that a scientist chooses to pose, and those problems that pose themselves. (He then went on to praise Paul Painlevé for having identified a problem that nature has been attempting to pose, and for having tackled it.) To sort out all problems of Nature in this fashion is not an assignment that a prudent scientist will want to undertake. Besides, a scientist gains no merit by contemplating problems which his tools are too feeble to solve. Nevertheless, Poincaré's distinction is intuitively an essential one. Does a new scientific enterprise contrive new problems for the pleasure of solving them easily? Or does it, however short it may remain from achieving perfection, contribute to sharpening and understanding some problems that are very old.

First example. To try and answer these questions in the case of fractal geometry, the best is, of course, to first allude to the mountains, the clouds, the water eddies and the trees. To the highest degree, the problems raised by their geometry deserve to be described as being 'problems that pose themselves'. Man-the-artist must have been pondering them forever, but Man-the-scientist did not know how to start tackling them. (At this point of the argument, an etymological digression comes to mind. The root of *geometry* is the Greek γεωμετρία. The customary translation, 'measure of the Earth', used to suggest to us that 'geometry' had

once denoted the problems we now tackle by fractals. But we are no longer sure; did it denote a far-reaching 'measure of the Earth', or a practical 'measure of the land' (as claimed in Eric Partridge's *Origins*). The *Oxford English Dictionary* points out that 'in early quotations, geometry is chiefly regarded as a practical art of measuring and planning, and is mainly associated with architecture.')

Second example. We continue with quotes from the King James Version of the Bible.

> ... were all the fountains of the great deep broken up, and the windows of heaven were opened. And the rain was upon the earth forty days and forty nights. *Genesis* **6**, 11–12.
>
> ... there came seven years of great plenty throughout the land of Egypt. And there shall arise after them seven years of famine... *Genesis* **41**, 29–30.

Awe of the actual unpredictability of the weather obviously permeates the stories of Noah's Flood, and of Joseph son of Jacob, the interpreter of Pharaoh's dream of the Seven Fat and Seven Lean Cows. More down-to-earth records confirm that such occurrences are in fact common in meteorology. On the other hand, none of the standard tools of modelling can account for these symptoms. Arguing that this failure could not be 'fixed' by small changes led us, many years ago, to draw a certain sharp distinction, to coin for it a flippant but immediately useful terminology, and to introduce very different tools that eventually become integrated in fractal geometry.

When a natural phenomenon is such that its action is felt much of the time, yet most of its effects concentrate 'oligopolistically' in one or a few largest events, the phenomenon is now said to obey the *Noah Effect*. A phenomenon is said to obey the *Joseph Effect* when it involves 'trends' of arbitrary slowness, whose cumulative effect exceeds the effect of even the largest addend.

We cannot dwell upon our studies of these two effects, beyond saying that the tools we used in these studies were eventually redesigned to yield our fractal models of mountains and of clouds.

Third example. It is 'turbulence'. To announce that at long last 'a cure' for it has been found seems a way of securing publication and an audience, but there is no cure as yet, only small advances raising big new questions. Turbulence can be taken as a prototype of the fractal phenomena that are old and interesting, and at the same time resistant to analysis. Several of the decisive early steps towards fractal geometry, such as the introduction of multifractal measures, were taken in 1968–76 when we were working on turbulence. Multifractal and fractal tools have contributed their share to describing and understanding turbulence, but it has not been explained, yet.

Fourth example. It is far less widely known than the preceding three, but far more relevant to the topic of the present meeting. It concerns *fractal growth* models and *fractal aggregates*. The discovery of diffusion limited aggregation (DLA) by Witten and Sander has initiated one of the most surprising and challenging quests of the last ten years in condensed matter physics. A complete understanding of the fractal and multifractal properties of such aggregates remains an open and intriguing problem, to which we shall return momentarily. Yet, one who sees a fractal aggregate for the first time is likely to experience a feeling of déjà-vu. Figure 2, plate 1, suggests one reason why.

FIGURE 2. Viscous fingers in a chunk of amber. This photograph by Paul A. Zahle, Ph.D. is © 1977, National Geographic Society. It has appeared in the *National Geographic Magazine* in September 1977, pp. 434–435. To quote from the accompanying text by T. J. O'Neill, 'apparently a tiny crack developed in an already solidified lump of resin; fresh, sticky resin then began to fill the narrow crevice. Air also crept along the fracture plane, thus forming the array of mosslike pseudo-fossils. Some foreign substance – perhaps iron oxide – [provided color]' ... 'The classical Greeks called [amber] *electron*.' ... 'Not until when the Roman author Pliny made public his *Historia Naturalis*, was amber scientifically described as a product of the plant world.' Not until fractal geometry had become available, could the 'array of mosslike pseudo fossils' be recognized as an example of viscous fingering, and could become an object for quantitative study.

Fractals and the hard-pressed engineer

Yet another reason to appreciate the language of fractal geometry is in some way a restatement of the third. The new language promises to be effective in engineering. Self-styled sophisticates tend to either forget or spurn the needs of the practical man. One reason is that he does not have the luxury of waiting until the phenomena he chooses or is asked to try and control have been explained to the satisfaction of the sophisticates. Instead, he finds himself lost in Galileo's 'dark labyrinth', where he does not even know what signs to look for or what to measure.

The list of investigations where fractals matter already or promise to matter very soon from the viewpoint of phenomenology and of engineering is already long, but to comment in turn on each 'case' would be tedious and would generate little light. Let us, instead, give a few typical examples.

Rock-bottom, a prerequisite of engineering, and also a continuing goal of science, is to *describe* nature quantitatively. But everyone who has tried knows that *to see* is a skill one must learn, and that one must learn what to measure. All too many disciplines harbour the strong wish of becoming quantitative, but do not know even how to begin. One standard way is to ask new questions for known answers, that is to borrow procedures from disciplines that have already reached a quantitative stage, and to hold on to these procedures if they appear to be effective. One finds that procedures one could borrow are not particularly numerous. While the diversity of nature appears to be without bound, the number of techniques one can use to grasp nature is extremely small and increases very rarely. Therefore, the enthusiasm usually generated by the birth of a new technique and the desire to test it more widely is healthy, and must not be disparaged. In a number of notable cases, the new fractal additions to Galileo's geometric alphabet prove to be of great help in efforts to see and to measure. Let us examine a few cases.

Viscous flow through porous media, e.g. the flow of water pushing oil, has proven recently to admit to several régimes, one of which is a 'front', which is an effective and desirable configuration, and another one is 'fractal fingering', which is undesirable. After this range of possibilities had become known, a colleague of ours saw a very old article on viscous flow. Next to a photograph that could have been taken yesterday, a diagram meant to summarize what the original photographer had seen in his work. Unfortunately, what he had seen turned out to lead nowhere, but what he has smoothed away turned out to prove important, and it included the fractal features.

Our second example is from hydrology. There is no question that the design of aqueducts and of dams involves many aspects of the science of materials. But what about the variability of the water discharge in rivers? Even if a full climatological explanation were available for the long-run component of this very erratic process, tasks have been assigned to different professions in such a way that the availability, or lack of availability, of a climatological explanation cannot possibly matter to the water resources community. Yet, when, starting in 1963, we advanced a model of the long-run persistance in river discharges, we found that explanation was perceived as mattering very much. Our model having an infinite

memory span. The first and most frequent questions were the following: 'Why choose so peculiar a model?' 'Has this model already been seasoned in the usual way, by being used in physics?' and 'What is the climatological explanation of this model?' In the water resources community, the most quantitatively inclined practitioners seemed intimidated by assertions of the primacy of explanation over everything else, afraid perhaps of hearing someone thunder 'But where is the science behind what you do?' We argued that it is best to perform each task in its time. The irony is, of course, that within a few years our model ceased altogether to be 'peculiar', because close counterparts were discovered, and soon adopted, in many chapters of 'mainstream' physics. These counterparts, as well as our model, have become building blocks of fractal geometry.

Our third example is from economics. To elaborate further upon the difference we see between the roles of fractals in engineering and in science, let us mention that security and commodity prices had been in the early sixties the topic of the first descriptive account that was to be later counted as fractal. It is a widespread assumption that price is a continuous and differentiable function of time. We claimed not only that it is not obvious and not only that it is contrary to the evidence, but that it is in fact contrary to what should be. The reason is that a competitive price should respond, in part to changes in anticipation, which can be subject to arbitrarily large discontinuities. (What happened on the Stock Exchanges on 19 October 1987 comes first to mind in 1988, but in the early 1960s such examples were viewed as things of the past.) On the basis of discontinuity combined with suitable self-similarity, we proposed a model of price variation, which had to incorporate the property of infinite variance of price increments. Our study of prices kept eliciting the comments already mentioned in the context of hydrology, the third one being phrased as 'Your models look fine, but how do you relate them to economic theory?' In moments of irritation, we are quoted are responding 'There is, as yet, no explanation for these findings; in fact *no* explanation could reasonably be expected to come from existing economic theory. After all, this theory has been growing for well over a century, and has yet to predict anything.'

The preceding sections show how difficult and unpopular is the task of defending the worth of scientific investigations that have not 'risen above' phenomenology. This difficulty is not new, in fact is very familiar in everyday life, where one can and must deal honestly with imperfection, while both preserving and keeping in check the dream of a More Heavenly City. In the context of fractal geometry, the Heavenly City of Explanation has already been reached in several cases, of which notable examples will be given. Everyone should rejoice, but it must immediately be acknowledged that other cases remain more like everyday life.

The scientific aspects of fractals are best discussed in two groups, those that are the most highly developed, and those which remain the most challenging.

The role of fractals in the theoretical sciences

The most highly developed examples of this role divide naturally into those which involve chance, and those which do not. The former are best exemplified by a chapter of statistical physics, called percolation theory, in which fractals have

risen highest above rock-bottom phenomenology. In this chapter, the fractal description is admirably complete, and the physics have been shown to be ruled by geometry, by a small number of quantities, each of them the fractal dimension of some specified portion of a geometric shape called critical percolation cluster. Furthermore, the basic dimensions have been deduced from basic physics, and some even turn out to be rational numbers! The reduction of physics to geometry being one of the basic goals of physics, the role of fractals in percolation theory is close to perfection, even though the mathematician will interject that many basic facts that the physicist holds true have not been proven in full rigour, at this point. Most unfortunately, however, percolation theory is well outside the main highways of mathematical physics, so that it is best to move on.

A second source of completely understood random fractals is found in probability theory, namely in the sets of measures that illustrate limit theorems concerning the sums or products of random effects. Brownian motion is the prime example, one that has preceded fractal geometry by decades but has been incorporated in its fold. The Brown surfaces we have used as models of relief are another example. Self-avoiding random walk also shows every sign of being a fractal, though (again) the fact has not been proven in full rigour. The basic multifractal measures are obtained as limits of products of random factors.

Among non-random fractals of immediate relevance to physics, the best understood ones result from the fact that 'basins of atttraction' are usually bounded by fractal sets. Let us give an intuitive example, and then amplify this statement. Since a drop of water that falls on the United States and does not evaporate will eventually end in either the Atlantic or the Pacific, one can say that each of the two Oceans has a basin of attraction. The two basins are bounded by the Continental Divide, which our fractal model of relief happens to represent usefully by a fractal curve. More formally, consider a dynamical system that eventually converges to either of several limit states. Each limit state has a basin of attraction, and the typical situation is when the boundaries between the basins are fractal sets. The first example was advanced by Pierre Fatou, in the work that was already mentioned when we discussed the impact of fractals upon mathematics. In this case, which is why all the basins of attraction taken together are now called Fatou set, the boundaries of these basins being called Julia sets. The broad impact of the old Fatou and Julia work was felt only after it had been made part of fractal geometry. From the viewpoint of physics, however, the most important recent development may well be the proof that basin boundaries are also fractal in more general maps in continuous time.

Fractals and the great equations of mathematical physics

One can, last but not least, appreciate the language of fractals because it has already been used by master physicists and other scientists to produce beautiful works that add to our understanding of how the world is put together, and in particular of our understanding of the extent to which there is truth in the widely held notion that geometry rules physics.

What is generally perceived as the highest level of natural science is the study of the great old equations of mathematical physics: those of Euler, Laplace,

Fourier and Navier–Stokes. The fact that these equations are *differential* implies a degree of smoothness that may seem *a priori* to forbid any connection with the rough and fragmented world of fractals. But it turns out that connections do exist, and that they are very important, even though some of them may have raised new questions in numbers exceeding the old questions they have already solved.

Our own contribution, which was the first interplay of fractals and differential equations, concerns the equations of fluid motion. We started with the old idea of Oseen, developed by J. Leray, that 'turbulence' is the name one gives to the effects of the singularities of Navier–Stokes equations. Granting this notion has allowed us to transform our fractal or multifractal models of turbulence into conjectures concerning the singularities of the equations. In specific mathematical terms, we introduced Hausdorff dimension as a new concern and a new tool in the study of the Navier–Stokes equations. The tool has proven attractive and the study to be fruitful, but by no means easy.

In the study of the Fourier equation, fractals have found a very attractive role in the notion of fractal fronts and of fractal foams of diffusion. The description we like best concerns a finite and discrete triangular lattice with a (hexagonal) ball placed on each lattice site. Let the abscissas of the balls run from $-x_{\max}$ to $+x_{\max}$, and start with the initial conditions where a ball is the colour of sand if its abscissa is negative, and is blue if the abscissa is positive. At each discrete instant of time, let 'couples' of neighbouring balls be chosen at random, independently of each other, and let their colours be interchanged. (The probability of being chosen is taken to be small, and if chance wants to choose a given ball as part of more than one 'couple', this ball can be left alone.) After a while, the relative proportion of blue balls will become a function of x, and the expected relative proportion is well known from the theory of diffusion of heat. Expectations, however, give an incomplete view of reality. What about the precise 'front' between the two colours? To define a front, reverse the colour in the sand islands entirely surrounded by blue sea, and in the blue lakes entirely surrounded by sand-coloured beach. This leaves us with a blue sea and a sand beach, separated by a shore line. Well, this shoreline happens to be a fractal curve, and its dimensions happens to be $D = \frac{7}{8}$. To be more precise, it does not becomes a fractal curve until the sites are down-scaled to infinitesimal sizes. As it wanders back and forth, this shoreline defines peninsulas that are attached to the beach by a single linking site. Imagine that these linking sites' colours are reversed. This will create a messy 'foam' separated from both beach and sea by wiggly curves: both wiggly boundaries happen to be fractal curves, and their dimension happens to be $D = \frac{4}{3}$. Higher-dimensional space diffusion creates an even more significant 'diffusion' foam. The proofs of the above assertions belong to the theory of percolation; hence, they are rigorous by the standards of physics, but continue to open purely mathematical problems.

The last but not least of the links between fractals and mainstream mathematical physics is raised by DLA and its many variants. It concerns the Laplace equations with interplay between, on the one hand, the solution of the equation at time t, and, on the other hand the displacement of the boundaries between times t and $t+1$. The body of this book, however, contains so much about DLA, that it is hardly worth dwelling on it. This new problem promises to be

difficult, and to stay with us for a long time. A conservative may even claim that it should have been mentioned in an earlier section concerned with phenom-emology, but this is a minor issue. It is, we think, a feather in a cap of fractal geometry that it has created in advance the environment and the tools that had allowed the study of DLA to move forth as quickly as it had done.

By design, this paper is devoid of complete references. Also, all names of living persons were meant to be avoided; that is, the few that have escaped us are not meant in the least to disparage the persons who fail to be named. Any mention of our own work, again, merely demonstrates a regrettable lack of consistency. For references, see the papers in this Symposium, as well as the following list of books. It was roughly up to date in February 1989, but probably fails to be fully complete.

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Additional note. An extensive, though far from complete, bibliography of fractals has been published as 'Resource Letter FR-1: Fractals', and a *Reprint book* has been announced by the American Association of Physics Teachers. The reference of the bibliography is Hurd, A. J. 1988 *Am. J. Phys.* **56**, 969–975.

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Discussion

A. Blumen (*Universität Bayreuth, F.R.G.*). Could Professor Mandelbrot please comment on the fact he mentions in one of his papers on multifractals, that 'each $\rho(\alpha)$ has its own multiplicative factor M'? Is there any signature of $\rho(\alpha)$, say in a similar way as we obtain the central limit theorem for certain distributions and Lévy-forms for others?

B. B. Mandelbrot. For sums of independent identically distributed addends $\log M$, the large deviations distribution $\rho(\alpha)$, hence the function $f(\alpha)$, is fully determined by the distribution of the addends $\log M$, and conversely. Thus, in the strict sense, there is no universality whatsoever. To the best of my knowledge (but I do not know all that much), the same is true when the addends are dependent and non-identical. This fact about the multifractals is very significant in their theory, and must be recognized. I think that the usual formalistic approach had hidden it.

But what does it mean in practice? It need not necessarily mean much. In particular, the tails of the empirical $f(\alpha)$s depend on high moments, and are determined with very low precision. It is tempting to define, for any given $\epsilon > 0$, a 'class of approximate ϵ-universality' that would include all the M that yield the same $f(\alpha)$ 'within ϵ'. (The strip of uncertainty could be either vertical or horizontal.) Such a 'neighbourhood of $f_0(\alpha)$' would contain $f(\alpha)$s for which the slope at α_{max} and α_{min} is infinite, and other $f(\alpha)$s for which the slope is finite. The corresponding class of ϵ-universality for M may, or may not, contain random variables that are not particularly alike in other ways. The topic is wide open. It deserves to be examined carefully, and are right to be concerned.